A result of existence and uniqueness for a cavity driven flow. Analytical expression of the solution.

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Abstract
In this work a result of existence and uniqueness for a plane cavity driven steady flow is deduced using an analytical method for the resolution of a linear partial differential problem on a triangular domain. The solution admits a symbolic expression based on integration over the domain. Some examples of flow are computed and graphed. In particular, it is shown a realistic example of a shear-forced flow with two eddies, usually computed only by numerical methods. The mathematical techniques used for the demonstration of the main result are elementary.

Keywords
incompressible flow, stream function, differential problem, vortices, existence and uniqueness of solution.

1 A theorem of existence and uniqueness

Let Ω be (the interior of) a triangular domain in $\mathbb{R}^2$, the cartesian $\{x, y\}$ plane, with vertices $O = (0, 0)$, $A = (2a, 0)$, $B = (a, a)$, where a is a positive real number. Note that the triangle OAB is rectangular and $\overline{OB} = \overline{BA}$. Let be $f(x, y) \in C^0(\overline{\Omega}, \mathbb{R})$. We want to resolve the differential problem

$$- \partial^2_{xx}\phi + \partial^2_{yy}\phi = f \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \partial\Omega \tag{1}$$

for a function $\phi(x, y)$, $\phi \in C^2(\overline{\Omega}, \mathbb{R})$. Note that the partial differential equation $-\partial^2_{xx}\phi + \partial^2_{yy}\phi = f$ admits a general solution of the form (see [5] or [10]) $\phi(x, y) = g(-x + y) + h(x + y) + \phi_0(x, y)$, where g and h are arbitrary real functions and ϕ_0 is a particular solution of the equation. But this general expression

is not very useful for applying the boundary condition $\phi = 0$ on $\partial\Omega$. It is more interesting and instructive the following direct method.

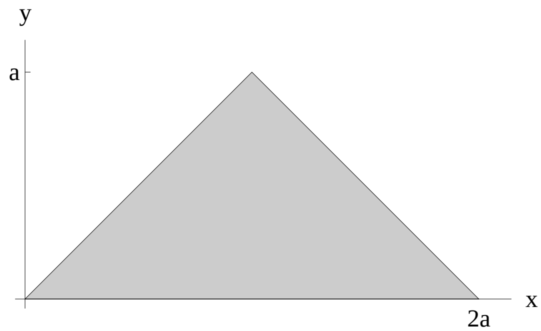

Figure 1: *The domain in the xy-plane.*

Consider the differential operator $-\partial^2_{xx} + \partial^2_{yy}$ written as $(\partial_x + \partial_y)(-\partial_x + \partial_y)$, and consider a linear transformation rule for cartesian coordinates $X = ax + by$, $Y = cx + dy$. If we want to have $2\partial_X = \partial_x + \partial_y$ and $2\partial_Y = -\partial_x + \partial_y$, using the chain rule it must be $a = b = 1 = d = 1$ and $c = -1$, that is

$$X = x + y, \quad Y = -x + y \tag{2}$$

The transformation is invertible:

$$2x = X - Y, \quad 2y = X + Y \tag{3}$$

With the notation $\Phi(X, Y) = \phi(x(X, Y), y(X, Y))$ and analogous for f, the differential equation $-\partial^2_{xx}\phi + \partial^2_{yy}\phi = f$ becomes

$$4\partial^2_{XY}\Phi(X, Y) = F(X, Y) \tag{4}$$

Note that the transformation (2) is a 45°-rotation and a $\sqrt{2}$-dilation of the plane $\{x, y\}$. Also, the boundary condition doesn't change: $\Phi = 0$ on $\partial\Omega$ (for simplicity we use for the domain in the plane $\{X, Y\}$ the same symbol Ω used for the plane $\{x, y\}$). For example, $\Phi(X, -X) = \phi(x, 0) = 0$. Therefore, the differential problem (1) becomes

$$4\partial^2_{XY}\Phi = F \quad \text{in } \Omega, \quad \Phi = 0 \quad \text{on } \partial\Omega \tag{5}$$

We remark the fact that the operator $4\partial^2_{XY}$ is the *canonical form* of the differential operator $-\partial^2_{xx}\phi + \partial^2_{yy}$, which has the lines $y = x$ and $y = -x$ as *characteristic curves* ([5] or [10]).

Now we want to discuss the resolution of the differential problem (5).

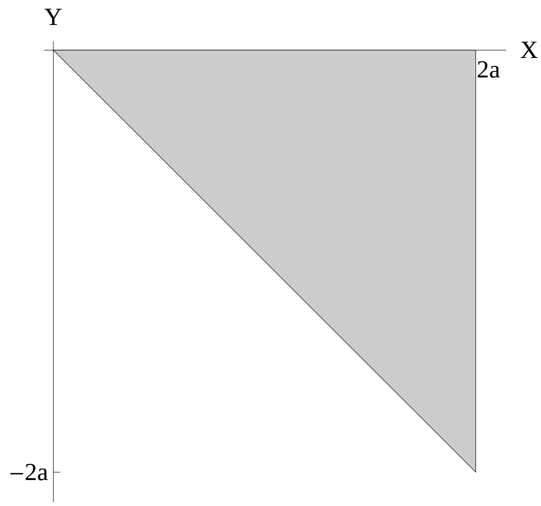

Figure 2: *The domain in the XY-plane.*

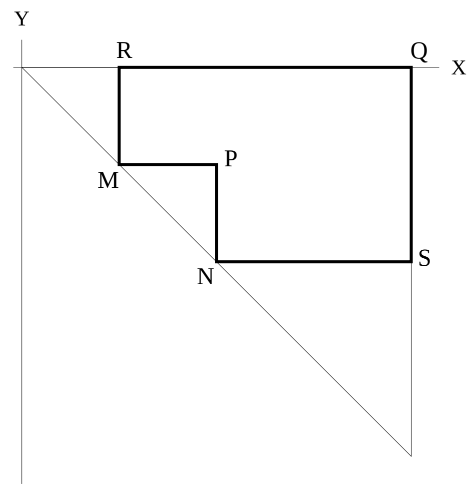

Figure 3: *Domain of integration.*

Let $P = (X, Y)$ be a point in the interior of the domain Ω. Then we can construct the polygon Σ using segments parallel to X and Y axes (see Fig.3). Note that $M = (-Y, Y)$, $R = (-Y, 0)$, $Q = (2a, 0)$, $S = (2a, -X)$, $N = (X, -X)$. Using the identity $2\partial^2_{XY} = (\partial_X\partial_Y + \partial_Y\partial_X)$, from the differential equation it follows that

$$2\int_{\Sigma}\left[\partial_X\partial_Y\Phi(X,Y)+\partial_Y\partial_X\Phi(X,Y)\right]dXdY=\int_{\Sigma}F(X,Y)dXdY \tag{6}$$

Now apply the Green theorem ([8] or [11]) to the first integral:

$$\int_{\Sigma}\left(\partial_X\partial_Y\Phi+\partial_Y\partial_X\Phi\right)dXdY=\int_{\partial\Sigma}\left(\partial_Y\Phi\ dY-\partial_X\Phi\ dX\right) \tag{7}$$

It is now simple to calculate the line integral along the edges of the polygon Σ (note that the boundary must be walked in counter-clockwise sense):

$$\begin{aligned}&\int_{\partial\Sigma}\left(\partial_Y\Phi\ dY-\partial_X\Phi\ dX\right)=\\&=-2\Phi(P)+2\Phi(N)-2\Phi(S)+2\Phi(Q)-2\Phi(R)+2\Phi(M)\end{aligned} \tag{8}$$

So we have

$$\begin{aligned}&-2\Phi(P)+2\Phi(N)-2\Phi(S)+2\Phi(Q)-2\Phi(R)+2\Phi(M)=\\&=\int_{\Sigma}FdXdY\end{aligned} \tag{9}$$

Now apply the boundary condition $\Phi_{|\partial\Omega}=0$: it follows that

$$\Phi(P)=\Phi(X,Y)=-\frac{1}{2}\int_{\Sigma(X,Y)}F(t,s)\ dt\ ds \tag{10}$$

with the consequence that, if the point $P(X,Y)$ lies on the boundary of Ω, that is if $P=M=N$ or $P=S$ or $P=R$, the function F must satisfy the necessary condition

$$0=\int_{\Sigma(X,Y)}F(t,s)\ dt\ ds\quad\forall(X,Y)\in\partial\Omega \tag{11}$$

It is easy to see that previous condition can be written in a more explicit fashion:

$$\int_{X}^{2a}\int_{-X}^{0}F(t,s)\ ds\ dt=0\quad\forall X\in[0,2a] \tag{12}$$

Therefore we have shown that a solution to differential problem (5), and hence to (1), exists if and only if F satisfies condition (12). Also, formula (10) is an analytical expression for a solution. Note that, denoted by T the point $(X,0)$, the integral can be divided into the two integrals defined on the two simple rectangles $PTRM$ and $NSQT$.

Now we discuss uniqueness of solution. Suppose to have two solutions Φ_1 and Φ_2 for the problem (5). Then $\Phi=\Phi_1-\Phi_2$ is a function such that $\partial^2_{XY}\Phi=0$ $\forall(X,Y)\in\Omega$ and $\Phi_{|\partial\Omega}=0$. Note that we can write

$$\partial_Y \left[\partial_X \Phi\right]^2 = 2\, \partial_X \Phi\, \partial^2_{XY} \Phi = 0 \tag{13}$$

Applying the Green theorem to domain Σ for the expression $\partial_Y \left[\partial_X \Phi\right]^2$, we have

$$0 = \int_{\partial\Sigma} \left[\partial_X \Phi\right]^2 dX =$$

$$= \int_N^S \left[\partial_X \Phi\right]^2 dX + \int_Q^R \left[\partial_X \Phi\right]^2 dX + \int_M^P \left[\partial_X \Phi\right]^2 dX \tag{14}$$

Using integration by parts, the following identity holds:

$$\int \left[\partial_X \Phi\right]^2 dX = \Phi\, \partial_X \Phi - \int \left[\Phi\, \partial^2_{XX} \Phi\right] dX \tag{15}$$

Hence, being $\Phi_{|\partial\Omega} = 0$, the second integral in (14) is null, therefore

$$\int_N^S \left[\partial_X \Phi\right]^2 dX + \int_M^P \left[\partial_X \Phi\right]^2 dX = 0 \tag{16}$$

The two integrals are evaluated in the same sense of the integration path, so that $\partial_X \Phi = 0$ along the segments NS and MP, therefore $\Phi(P) = \Phi(M)$. But $\Phi(M) = 0$, being $M \in \partial\Omega$, so for a generic point $P = (X, Y)$ we have $\Phi(P) = 0$. The solutions Φ_1 and Φ_2 are identical.

We have shown the following result (remember that $X = x+y$, $Y = -x+y$):

Theorem 1 *Let f be a real function of $C^0(\overline{\Omega}, \mathbb{R})$ such that*

$$\int_X^{2a} \int_{-X}^0 f\left(\frac{t-s}{2}, \frac{t+s}{2}\right)\, ds\, dt = 0 \quad \forall X \in [0, 2a] \tag{17}$$

Then the differential problem

$$-\,\partial^2_{xx}\phi + \partial^2_{yy}\phi = f \quad in\ \Omega, \quad \phi = 0 \quad on\ \partial\Omega \tag{18}$$

has one and only one solution in the space $C^2(\overline{\Omega}, \mathbb{R})$. The solution is given by the formula

$$\begin{aligned} \phi(x,y) = & -\frac{1}{2}\int_{x-y}^{x+y} \int_{-x+y}^0 f\left(\frac{t-s}{2}, \frac{t+s}{2}\right)\, ds\, dt - \\ & -\frac{1}{2}\int_{x+y}^{2a} \int_{-x-y}^0 f\left(\frac{t-s}{2}, \frac{t+s}{2}\right)\, ds\, dt \end{aligned} \tag{19}$$

2 An application: cavity driven flows

In this section we discuss an application of previous theorem to a problem of two-dimensional cavity driven flow, that is a plane flow confined in a cavity and induced by the stress due to a primary flow external to the cavity (see [9]). This phenomenon has great importance in scientific research (see e.g. [7]) and technological applications. Assume that the cavity has the shape of the triangle OAB of Fig.1 in the xy-plane. Stress due to the primary flow acts on the horizontal edge OA. We suppose that the fluid is newtonian and incompressible, that is plane stress $\mathbb{T}$ and plane strain-rate $\mathbb{D}$ tensors are linked by the formula (see [4])

$$\mathbb{T} = 2\mu\mathbb{D} \tag{20}$$

where μ is the dynamic viscosity and $2\mathbb{D}_{ij} = (\partial_j v_i + \partial_i v_j)$ (see [6]), where $\mathbf{v}$=(v_1, v_2)=(v_x, v_y) is the flow velocity field. Plane incompressible flows admit a stream function ([6]), that is a function $\Psi(x, y)$ such that

$$u = v_x = \partial_y \Psi, \quad v = v_y = -\partial_x \Psi \tag{21}$$

Therefore, a plane newtonian incompressible flow is described by the partial differential equation

$$-\partial^2_{xx}\Psi + \partial_{yy}\Psi = \frac{1}{\mu}\mathbb{T}_{xy} \tag{22}$$

In the next of the paper we suppose to know the analytical expression of $\mathbb{T}_{xy}$ and we try to find a solution of (22) for a stream function Ψ such that $\Psi_{|\partial\Omega} = 0$. This boundary condition is usual for plane incompressible flow (see [6] and [2]), but in the case of a cavity driven flow it (or an analogous $\Psi_{|\partial\Omega} = \text{const}$) has an important physical meaning. In fact, if $\Psi_{|\partial\Omega} = 0$, then $\partial\Omega$ is a level curve for Ψ, therefore at each point of the boundary $\nabla\Psi$ is orthogonal to the tangent of the boundary itself ([11]). But $\nabla\Psi = (-v, u)$, which is orthogonal to the flow velocity field (u, v). Therefore, at each point of $\partial\Omega$, the geometrical tangent and the velocity field are parallel, that is the flow is confined into the cavity Ω.
Applying theorem (1), it can be stated that if $\mathbb{T}_{xy} \in C^0(\overline{\Omega}, \mathbb{R})$, then there is a unique stream function $\Psi \in C^2(\overline{\Omega}, \mathbb{R})$ solving the linear equation (22) with boundary condition $\Psi|_{\partial\Omega} = 0$. Note that [1] consider a nonlinear problem about cavity driven flow where uniqueness can fail.

We consider at first the more simple analytical form for a possible stress:

$$\mathbb{T}_{xy} = \mu(c_1 y + c_2) \tag{23}$$

In this case, along the horizontal edge OA ($y = 0$) of the cavity the stress is constant. From theorem (1), a solution to our differential problem exists if the function $c_1 y + c_2$ satisfies the condition (17). It is easy to show that the condition is satisfied for all $X \in [0, 2a]$ if and only if $2c_2 = -ac_1$. Note that in this case the stress has expression

$$\mathbb{T}_{xy} = \mu c_2 \left(-\frac{2}{a} y + 1 \right) \tag{24}$$

and for $y = \frac{a}{2}$ it changes its sign. So flow can recirculate. We make the choice $c_2 = -8a$, so that $c_1 = 16$. The differential problem to solve is

$$-\partial^2_{xx}\phi + \partial^2_{yy}\phi = 16y - 8a \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \partial\Omega \tag{25}$$

From formula (19), using the transformation rule $X = x + y$, $Y = -x + y$ the solution to (25) can be computed by

$$\begin{aligned} \Psi(x,y) = \quad & - \quad 4\int_{x-y}^{x+y}\int_{-x+y}^{0} (t+s-a)\, ds\, dt - \\ & - \quad 4\int_{x+y}^{2a}\int_{-x-y}^{0} (t+s-a)\, ds\, dt \end{aligned} \tag{26}$$

which gives the expression

$$\Psi(x,y) = 2y^3 - 2x^2y - 4ay^2 + 4axy \tag{27}$$

for the stream function of the flow. The velocity field is $(\partial_y\Psi, -\partial_x\Psi) = (-2x^2 + 6y^2 + 4ax - 8ay, 4xy - 4ay)$. It is interesting to find the points where the velocity is null. Solving the algebraic system $(-2x^2 + 6y^2 + 4ax - 8ay, 4xy - 4ay) = (0,0)$, we find as expected the three vertices $(0,0)$, $(2a,0)$ and (a,a), and also the interior point $(\frac{a}{3}, a)$ which is the center of the recirculation gyre (see Fig.4).

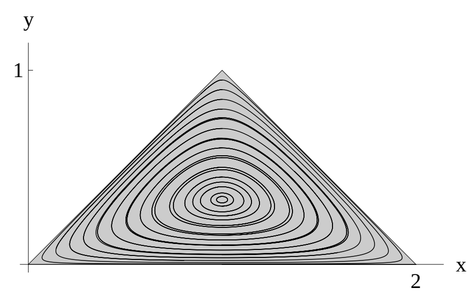

Figure 4: *Flow path-lines in the case* a=1.

Now we consider a more interesting case. Let the stress be described by a sinusoidal expression of the form

$$\mathbb{T}_{xy} = A\mu \cos(ky) \tag{28}$$

with A and k real numbers. Using (17), it is easy to show that if we suppose $q = 0$, then

$$k = m\frac{\pi}{a}, \quad m = 2n+1, \quad n \in \mathbb{N} \tag{29}$$

is the condition for existence and uniqueness of a flow in the triangular cavity. Consider $m = 1$. By integration (19), the analytic form of the stream function, solution of the differential problem, is

$$\Psi = -\frac{2Aa^2}{9\pi^2}\left[\cos\left(\frac{3\pi}{a}y\right) + \cos\left(\frac{3\pi}{2a}(x-y)\right) - 2\cos^2\left(\frac{3\pi}{4a}(x+y)\right)\right] \tag{30}$$

and Fig.5 shows some path lines, where one primary central eddy and three secondary eddies are present. It is also interesting to draw the graph of $u = \partial_y \Psi$ for $x = a$ and y variable in the range $[0, a]$: there are two values of y, not equal to a, for which $u = 0$. One of the two values is equal to the value where $u = 0$ in the previous case of the linear stress (see Fig.6).

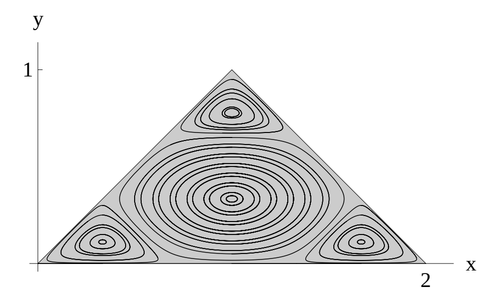

Figure 5: *Flow path-lines in the case of the sinusoidal stress, with* a=1, A=5.

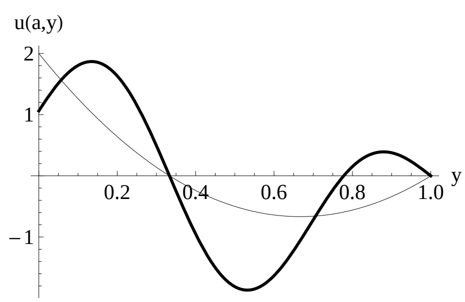

Figure 6: *Comparison of* $u = \partial_y \Psi$ *in the linear (thin line) and sinusoidal (thick line) case.*

A more realistic example is based on a stress with analytical expression of the form

$$\mathbb{T}_{xy} = \mu \sum_{m,n=0}^{4} a_{m,n}\, x^m y^n \tag{31}$$

Applying condition (17) for the computation of the coefficients $a_{m,n}$, a possible stream function is

$$\Psi(x,y) = (2y^3 - 2x^2y - 4ay^2 + 4axy)(y - 100x^2 - a)\left(y + \frac{1}{4}x - \frac{5}{6}a\right) \tag{32}$$

The horizontal component u of the velocity field, along the x-axes, is a 5-order x-polynomial whose graph is shown on Fig.7. The stress acts on the horizontal segment of the triangular domain as a variable shear of positive sign.

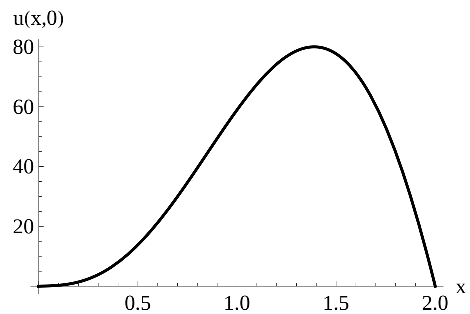

Figure 7: *Profile of* $u(x,0) = \partial_y \Psi(x,0)$ *in the case of a variable shear stress.*

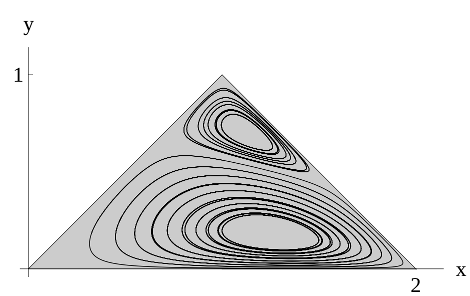

Figure 8: *Flow path-lines in the case of a variable shear stress, with* a=1.

The resulting flow path-lines (see Fig.8) show the presence of a primary gyre, and of a secondary gyre near the vertex opposite to the edge subjected to external stress. This image is similar to a corrisponding picture (fig.2(b)) in [2], where stream function is computed by numerical method.

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